From Foundational Physics to Dark Matter

Contents

1. From Mechanics to Dark Matter

2. From Gravitation to Dark Matter

From Mechanics to Dark Matter

Newton laws and Mechanics

The Newton laws is the following:

$$F = \frac{Gm_1m_2}{r^2} \tag{1}$$

The centripetal force in the circular motion is:

$$F = \frac{mv^2}{r} \tag{2}$$

Equating the gravitational force with the centripetal force:

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \tag{3}$$

Simplifying, we obtain the orbital velocity [9]:

$$v^2 = \frac{GM}{r} \rightarrow v(r) = \sqrt{\frac{GM(r)}{r}} \tag{4}$$

Kepler law

The third Kepler law for the planets motion in the Solar System:

$$T^2 \sim r^3 \tag{5}$$

where T is the orbital period and r is the semi-major axis of the orbit, replacing T formula in the Kepler law, we obtain:

$$T = \frac{2\pi r}{v(r)} \rightarrow (\frac{2\pi r}{v(r)})^2 \sim r^3 \tag{6}$$

With an approximation, the velocity decreases with the radius growth, indeed distant planets are slower in their orbital motion:

$$\frac{4\pi^2 r}{v(r)^2} \sim r^3 \rightarrow \frac{1}{v(r)^2} \sim r \rightarrow v(r) \sim \frac{1}{\sqrt{r}} \tag{7}$$

Conservation of angular momentum

The angular momentum L of an orbiting particle:

$$L = mvr \tag{8}$$

The conservation of angular momentum implies that if there is not the influence of external forces, except the central gravity, orbits will be stable and the angular momentum constant [10]:

$$L = const \rightarrow v \sim \frac{1}{r} \tag{9}$$

Conservation of energy

The total energy is the sum of the kinetic and potential energies:

$$E = K + U \rightarrow E = \frac{1}{2}mv^2 - \frac{GMm}{r} \tag{10}$$

The kinetic energy is:

$$K = \frac{1}{2}mv^2 \tag{11}$$

and the gravitational potential energy is:

$$U = -\frac{GMm}{r} \tag{12}$$

Applying the energy conservation, if the mass is concentrated in the center, the potential energy dominates at great distance and the velocity decreases with the radius. Consequently, stars will slow their motion at larger radii without additional mass. If the energy is negative the stars will be gravitationally bound, if the energy is positive the stars will escape.

Predictions

If all the mass is concentrated in the center, at large radii the velocity will decrease as the Keplerian behaviour: $v(r) \sim \frac{1}{\sqrt{r}}$ and the mass will be constant. In the Solar System distant planets are slower than nearest and in the same way distant stars will orbit with smaller velocity.

Observations

- Rubin and Ford measured the light spectrum of stars and gas in the galaxies [13].
- For the Doppler effect, if stars or gas is moving to the observer the light will shift to blue (blueshift), while if it is moving away from the observer the light will shift to red (redshift).
- Observing the gas and stars distribution at different radii from the galactic center, we can reconstruct the orbital velocity related to the distance. In most of spiral galaxies the velocities are constant also far from the center. The result is a flat rotational velocity curve beyond the optical radius (luminous visible region) [8].
- The constant velocity yields an increasing of the mass with the radius growth $M(r) \sim r$ and a stronger gravitational attraction that keeps the stars bound to orbits with high speed.
- If the stars in outer regions have high velocity their energy will be positive and with only the visible mass will escape from the galaxy.
- From the conservation laws of energy and angular momentum, we deduce the need of extra mass in order to increase the gravitational attraction.

From Gravitation to Dark Matter

Gravitational potential energy

From the Newton gravitational law, the gravitational potential energy is derived for a system of galaxies:

$$F = G\frac{m_1 m_2}{r^2} \rightarrow U = -G\sum_{i,j}\frac{m_i m_j}{r_{ij}} \tag{13}$$

If the object is spherical and has the physical dimension R and α is a numerical factor that measures how strongly the object's mass is concentrated towards its center, then the total gravitational potential energy will be of the form:

$$U = -\alpha\frac{GM^2}{R} \tag{14}$$

where the negative sign is related to the attractive gravity, $M = \sum_i M_i$, $R \rightarrow 0$, $U \rightarrow 0^-$.

Kinetic energy

Selecting the galaxies spectrum with the same redshift,computing the average redshift z_{cl}, we obtain the peculiar velocity v_i of each galaxy [7, 11].

$$v_i \sim c\frac{z_i - z_{cl}}{1 + z_{cl}} \sim c\Delta z \tag{15}$$

The kinetic energy for a system is the following expression [2, 12, 5, 15]:

$$K = \frac{1}{2}\sum_i m_i v_i^2 \tag{16}$$

Equating the kinetic energy with the gravitational potential energy:

$$\frac{1}{2}\sum_i m_i v_i^2 = \frac{GM^2}{R} \xrightarrow{x2} \sum_i m_i v_i^2 = \frac{1}{2}\frac{GM^2}{R} \rightarrow K = \frac{1}{2}U \tag{17}$$

Momentum of inertia

Recalling the Newton law for each particle of the system: $F_i = m_i a_i$, we can compute the inertial momentum for the spatial distribution time-related [15]:

$$I(t) = \sum_i m_i r_i^2(t) \tag{18}$$

The first derivative $\dot{r_i} = v_i$ is obtained:

$$\frac{dI}{dt} = 2 \sum_i m_i r_i \dot{r_i} \tag{19}$$

The second derivative $\ddot{r_i} = a_i$ and $a_i = \frac{F_i}{m_i}$ is the following:

$$\frac{d^2 I}{dt^2} = 2 \sum_i m_i (\dot{r_i} \dot{r_i} + r_i \ddot{r_i}) = 2 \sum_i m_i v_i^2 + 2 \sum_i m_i r_i a_i = 2 \sum_i m_i v_i^2 + 2 \sum_i r_i F_i \tag{20}$$

Introducing K and U

Replacing with the kinetic energy formula $K = \frac{1}{2}\sum_i m_i v_i^2$ the first term, we found the general form:

$$\frac{d^2I}{dt^2} = 4K + 2\sum_i r_i F_i \xrightarrow{/2} \frac{1}{2}\frac{d^2I}{dt^2} = 2K + \sum r_i F_i \tag{21}$$

In the gravitational case for homogeneous potential of degree -1 $U = \sum_i r_i F_i$, where the U is negative derived from:

$$F_{ij} = -G\frac{m_i m_j}{r_{ij}^3}(r_i - r_j) \rightarrow F_i = \sum_{ij} F_{ij} \tag{22}$$

From the internal gravitation force of two bodies:

$$\sum_i r_i F_i = \frac{1}{2}\sum_{ij}(r_i - r_j)F_{ij} = \frac{1}{2}\sum_{ij}(-G\frac{m_i m_j}{r_{ij}}) = U \tag{23}$$

Gravitational bound system

Considering the kinetic energy $\sum_i m_i v_i^2 = 2K$ and the gravitational potential $\sum_i r_i F_i = U$:

$$\frac{d^2 I}{dt^2} = 2\sum_i m_i v_i^2 + 2\sum_i r_i F_i = 2K + 2U \rightarrow \frac{1}{2}\frac{d^2 I}{dt^2} = K + \frac{1}{2}U = 2K + U \tag{24}$$

- For a stable and gravitational bound system $\frac{d^2 I}{dt^2} = 0 \rightarrow 2K + U = 0$.
- A system is gravitational bound if the total energy is negative $E = K + U < 0$ or in our case if the galaxies confined by gravity cannot escape in the space, indeed clusters cannot disperse and are gravitational bound.
- If $\frac{d^2 I}{dt^2} \neq 0$, the system can expand with an increasing momentum of inertia, or collapse with a decreasing momentum of inertia, but a cluster has a stationary distribution that cannot expand to infinity and cannot collapse [6].
- For a semi-stationary equilibrium we consider the temporal mean: $< \frac{d^2 I}{dt^2} > \sim 0$, where $I \neq 0$, positive and finite because it measure the distribution of galaxies.

Virial theorem

Considering a single light particle in circular orbit around a heavy one, the potential energy is:

$$U = -G\frac{Mm}{R} \tag{25}$$

Equating the gravitational force to the centrifugal force, in a circular orbit they counteract each other perfectly:

$$\frac{GmM}{R^2} = \frac{mv^2}{R} \tag{26}$$

The kinetic energy of the heavy particle is negligible, while the one of the light particle is:

$$K = \frac{mv^2}{2} = \frac{GmM}{2R} = -\frac{U}{2} \tag{27}$$

The virial theorem relates the time-averaged total kinetic energy of a stable, bound system to the time-averaged total potential energy [1, 14, 3]:

$$2 < K > + < U >= 0 \rightarrow < K >= -\frac{1}{2} < U > \tag{28}$$

Luminous vs dark matter

- The visible matter is composed of stars,gas and dust. Star mass is measured from the total luminosity of galaxies using stellar population models, while the gas mass is determined through the emission lines in infrared,radio,$H\alpha$ and CO.
- The sum of star and gas masses is not enough to reach the total mass, but reach only $10 - 20\%$.
- The stellar light decreases exponentially with the radius and the density of luminous matter decreases away from the center. If the mass is all luminous, it will saturates and the curve should decrease, but evidence shows that it increases.
- Zwicky found that the dynamic mass of the Coma cluster was higher than the sum of the masses of its single galaxies. Consequently, we need some additional mass distributed beyond the luminous regions[16].

Virial mass

From the relation between kinetic energy and gravitational potential, we can compute the total o virial mass:

$$\sum_i m_i v_i^2 = \frac{GM^2}{R} \rightarrow M^2 = \frac{R\sum_i m_i v_i^2}{G} \tag{29}$$

In this case the mass $M = \sum_i m_i$ is the sum of the mass of all N galaxies, if we consider galaxies with the same mass, $M = Nm$

$$\sum_i m v_i^2 = \frac{G(Nm)^2}{R} \xrightarrow{/m} \sum_i v_i^2 = G\frac{N^2 m}{R} \xrightarrow{xR,G} M \sim Nm = \frac{R\sum_i v_i^2}{G} \tag{30}$$

In a spherical cluster the velocity distribution is isotropic,the 3D velocity dispersion of galaxies is: $\sigma_{3D}^2 = 3 < (v_i - v_s)^2 >= 3\sigma_v^2$ and the total virial mass of the cluster is:

$$M_{vir} = \frac{3r\sigma_v^2}{G} \tag{31}$$

where $v_s = \frac{\sum_i v_i}{N}$ with N galaxies in the cluster. [4, 5, 12]

References I

[1] J. Baez. *The Virial Theorem Made Easy*. Virial theorem example from classical laws. 2020. URL: https://math.ucr.edu/home/baez/virial.html.

[2] Neta A. Bahcall and Craig L. Sarazin. "Cluster Velocity Dispersions and Masses". In: *The Astrophysical Journal* 258 (1982). virial of clusters and application with velocity dispersion, pp. 1–9. DOI: 10.1086/160050.

[3] James Binney and Scott Tremaine. *Galactic Dynamics*. 2nd. Cap. 4, sez. 4.3: virial theorem for auto-gravitational system. Princeton University Press, 2008. ISBN: 978-0-691-13026-2.

[4] R.G. Carlberg et al. "Galaxy Cluster Virial Masses and Omega". In: *apj* 462 (May 1996), p. 32. DOI: 10.1086/177125. arXiv: astro-ph/9509034 [astro-ph].

References II

[5] D. Cline. *The virial theorem*. Virial theorem for a system of particles. URL: https://phys.libretexts.org/Bookshelves/Classical_Mechanics/Variational_Principles_in_Classical_Mechanics_(Cline)/023A_Review_of_Newtonian_Mechanics/2.113A_Virial_Theorem.

[6] Peter Coles. *The Routledge Critical Dictionary of the New Cosmology*. https://ned.ipac.caltech.edu/level5/Glossary/Essay_virial. New York: Routledge, 1999. ISBN: 978-0-415-15915-8.

[7] D. W. Hogg. "Distance measures in cosmology". In: *arXiv:astro-ph/9905116* (1999). Sec 2.3: relation redshift and radial velocity. URL: https://arxiv.org/abs/astro-ph/9905116.

[8] Malcolm S. Longair. *Galaxy Formation*. 2nd. Cap. 2: galaxy dynamics; Cap. 5: evidence of dark matter and rotational curve. Springer, 2008. ISBN: 978-3-540-73478-4.

References III

[9] Gayatri Mohan and Umananda Dev Goswami. "Galactic rotation curves of spiral galaxies and dark matter in f(R,T) gravity theory". In: *International Journal of Geometric Methods in Modern Physics* 21.04 (Dec. 2023). ISSN: 1793-6977. DOI: 10.1142/s0219887824500828. URL: http://dx.doi.org/10.1142/S0219887824500828.

[10] Pedrosa, Susana E. and Tissera, Patricia B. "Angular momentum evolution for galaxies in a Lambda-CDM scenario". In: *AA* 584 (2015), A43. DOI: 10.1051/0004-6361/201526440. URL: https://doi.org/10.1051/0004-6361/201526440.

[11] P. J. E. Peebles. *Principles of Physical Cosmology*. 1st. Cap 9: Peculiar Velocities. Princeton, NJ: Princeton University Press, 1993.

References IV

[12] M. Richmond. *Using the virial theorem:mass of a globular cluster*. Virial theorem with velocity dispersion in kinetic energy. URL: https://phys.libretexts.org/Bookshelves/Astronomy__Cosmology/Supplemental_Modules_28Astronomy_and_Cosmology29/Cosmology/Astrophysics_28Richmond29/173A_Using_the_Virial_Theorem_-_Mass_of_a_Globular_Cluster.

[13] Vera C. Rubin and W. Kent Ford Jr. "Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions". In: *The Astrophysical Journal* 159 (1970). rotational velocity curve, p. 379. DOI: 10.1086/150317.

[14] Lyman Spitzer. *Dynamical Evolution of Globular Clusters*. Cap. 1: virial theorem for stellar systems. Princeton University Press, 1987. ISBN: 978-0-691-08549-4.

[15] Wikipedia. *The Virial Theorem Made Easy*. Virial theorem. URL: https://en.wikipedia.org/wiki/Virial_theorem.

References V

[16] Fritz Zwicky. “Die Rotverschiebung von extragalaktischen Nebeln”. In: *Helvetica Physica Acta* 6 (1933). Using virial theorem for clusters of galaxies and dark matter evidence, pp. 110–127.